Math 304 (Spring 2015) - Homework 4

Please note that this homework has two pages. There are 5 problems in total.

Problem 1.

Determine whether the following sets are subspaces of $\mathbb{R}^2$. Explain why!

(1) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + x_2 = 0 \right\}$

(2) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 x_2 = 0 \right\}$

(3) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + x_2 = 1 \right\}$

(4) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1^2 + x_2^2 = 1 \right\}$

Solution:

(1) Yes, it is a subspace. (I skip the details).

(2) No.

(3) No.

(4) No.

Problem 2.

Let us denote by $M_{2\times 2}(\mathbb{R})$ the set of all (2×2) matrices whose entries are all real numbers. In the textbook, $M_{2\times 2}(\mathbb{R})$ is also denoted by $\mathbb{R}^{2\times 2}$. We know that $M_{2\times 2}(\mathbb{R})$ is a vector space. Now determine whether the following sets are subspaces of $M_{2\times 2}(\mathbb{R})$. Explain why!

(1) The set of all (2×2) upper triangular matrices.

(2) The set of all (2×2) nonsingular matrices.

(3) The set of all (2×2) symmetric matrices.

(4) The set of all (2×2) matrices with determinant equal to 1.

Solution: I skip the details.

(1) Yes.

(2) No.

(3) Yes.

(4) No.

Problem 3.

(a) Let $\mathbb{P}_3$ be the vector space of all polynomials with degree less than or equal to 3. Determine whether the following sets are subspaces of $\mathbb{P}_3$.

(1) The set of all polynomials $p(x)$ in $\mathbb{P}_3$ such that $p(0) = 0$.

(2) The set of all polynomials $p(x)$ in $\mathbb{P}_3$ such that $p(0) = 1$.

(b) Let $C[-\pi, \pi]$ be the vector space of all continuous functions on the closed interval $[-\pi, \pi]$. Determine whether the following sets are subspaces of $C[-\pi, \pi]$.

(1) The set of all *odd* functions in $C[-\pi, \pi]$.

(2) The set of functions $f(x)$ in $C[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$.

Solution:

(a) (1) Yes.

(2) No.

(b) (1) Yes.

(2) Yes.

Problem 4.

Determine whether the following vectors form a spanning set of $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

Solution: We need to determine whether every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ of $\mathbb{R}^3$ is a linear combination of

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

Consider the linear system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 3 & 2 & 4 & b \\ 2 & 3 & 3 & c \end{array}\right]$$

The coefficient matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 4 \\ 2 & 3 & 3 \end{pmatrix}$$

is nonsingular (you need to give details here), so the system always has a unique solution. So

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

span $\mathbb{R}^3$.

Problem 5.

Find the null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix}$$

Solution: The row echelon form of A is

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the null space of A is

$$\alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$